On Surfaces Containing Two Pencils of Cubic Curves.

By C. H. Sisam.

1. In a recent article in the American Journal of Mathematics, Vol. XLI, p. 49, the author has discussed the surfaces generated by an algebraic system of cubic curves that do not constitute a pencil. In this article he studies the surfaces which contain two or more pencils of cubic curves.

2. Let m be the order of a given surface F which is generated by two pencils Σ_1 and Σ_2 of cubics which intersect in k variable points. We shall show that

$$mk \leq 18.$$

Any surface of order kx such that $kx > m$ contains all the curves of Σ_2, and hence has F as a component, if it contains $3x+1$ generic curves of Σ_1. Hence it contains F if it contains $3kx+1$ generic points on each of $3x+1$ generic curves of Σ_1, that is, if it satisfies at most, $(3x+1)(3kx+1)$ independent linear conditions. Then this number can not be less than the actual number of conditions that a surface of order kx must satisfy in order to contain F as a component, that is

$$(3x+1)(3kx+1) \geq \frac{(kx+3)!}{3!\,kx!} - \frac{(kx-m+3)!}{3!\,(kx-m)!},$$

or

$$3k(18-mk)x^2 + (3km^2-12km+18k+18)x - (m^3-6m^2+11m-6) \geq 0.$$

Since this inequality holds for all integer values of x such that $xk > m$, we must have $18-mk \geq 0$.

We have supposed, in the above proof, that F lies in three dimensions. But the theorem still holds if F belongs to $r > 3$ dimensions, since such a surface can be projected, without changing m or k, into one belonging to three dimensions.

Case I. The Cubics of Both Given Systems are Rational.

3. In this case the surface F is rational since it contains two pencils of rational curves. The entities Σ_1 and Σ_2, whose elements are the curves of the

two systems, are also rational since each defines a rational involution of order k on a generic curve of the other pencil.

$k=1$.

4. Let the pencils Σ_1 and Σ_2 be put in (1, 1) correspondence with pencils of lines in a plane Π having vertices at P_1 and P_2 respectively. Then the surface F and the plane Π are in (1, 1) correspondence in such a way that corresponding points are at the intersections of corresponding curves. Let μ be the order of the curves in Π that correspond to the hyperplane sections of F. Since these curves have points of multiplicity $\mu-3$ at P and P_1' we have $2(u-3)\leq\mu$ or $\mu\leq 6$.

5. The system of sextic curves in Π which have triple points at P and P' define parametrically a surface F which is of the given type. Any other such surface is a projection of this one, since any linear system of curves of order μ having $(\mu-3)$-fold points at P and P' is contained in the linear system of sextics. We conclude that if the curves of Σ_1 and Σ_2 are rational and $k=1$, *the surface is of order* 18, *belongs to an* S_{15}* *and is defined parametrically by the sextics in a plane which have two basis triple points or it is a projection (not necessarily from external points) of such a surface.*

6. If F belongs to S_{15}, the genus of a generic hyperplane section is equal to 4, that is, to the genus of a generic sextic with two triple points. It contains no cubic curves other than those of the two given systems.

$k=2$.

7. Let F be birationally transformed into the plane Π in such a way that the cubics of Σ_2 correspond to the lines through P'. Then the cubics of Σ_1 correspond to rational curves of order ν which have a $(\nu-2)$-fold point at P'. By a suitable birational transformation the order of the curves of this pencil can be reduced to 2. Let, then, $\nu=2$ and let P_1, P_2, P_3, P_4 be the basis points of this pencil of conics. To the hyperplane sections of F correspond curves of order μ which have a $(\mu-3)$-fold point at P' and multiplicities ρ_i at P_i, $(i=1, 2, 3, 4)$ such that $2\mu-(\rho_1+\rho_2+\rho_3+\rho_4)=3$. By a suitable birational transformation, such a system of curves can be reduced to one of the following three types:

(1) $\mu=4 \quad \rho_1=2 \quad \rho_2=\rho_3=\rho_4=1.$
(2) $\mu=3 \quad \rho_1=0 \quad \rho_2=\rho_3=\rho_4=1.$
(3) $\mu=3 \quad \rho_1=2 \quad \rho_2=1,\ \rho_3=\rho_4=0.$

* The symbol S_r denotes a space of r dimensions.

8. The surfaces defined parametrically by systems of types (2) and (3) are projections of special cases of surfaces of type (1) such that the latter surface has one or more double points. Hence, if the cubics of Σ_1 and Σ_2 are rational and $k=2$, *the surface is of order* 8, *belongs to an* S_7 *and is defined parametrically by the quartics in a plane which have a double and four simple basis points, or it is a projection of such a surface.*

9. If F belongs to S_7, its generic hyperplane sections are of genus 2. It contains four pairs of pencils of cubics such that cubics of opposite systems intersect in two points. One pencil of such a pair is defined by the pencil of conics through the double and three simple basis points. The other pencil is defined by the lines through the remaining basis points. It contains no other cubic curves.

$k=3$.

10. As in the preceding case, this surface can be transformed into a plane Π in such a way that the cubics of Σ_2 correspond to the lines through P' and the cubics of Σ_1 to the cubics which have a basis double point P_1 and five basis simple points $P_2, P_3, \ldots, P_6$. To the hyperplane sections of F correspond curves of order μ which have a $(\mu-3)$-fold point at P' and multiplicities ρ_i at P_i, $(i=1, 2, 3, \ldots, 6)$ such that $3\mu-(2\rho_1+\rho_2+\rho_3+\rho_4+\rho_5+\rho_6)=3$. If the order m of F exceeds three, these curves can be transformed into quartics which have a double point at P_1 and pass simply through $P_2, P_3, \ldots, P_6$ and P'. If $m\leq 3$ the surface is a projection of one with one or more singular points defined by a system of the above type. Hence, if the cubics of Σ_1 and Σ_2 are rational and if $k=3$, *the surface is of order* 6, *it belongs to an* S_5 *and is defined parametrically by the quartics in a plane which have a double and six simple basis points, or it is a projection of such a surface.*

11. There are thirty-two pencils of cubic curves on such a surface F_1 defined by the pencils of curves of order $i\leq 3$ in Π which have an $(i-1)$-fold point at P_1 and pass through $2i-1$ simple basis points. The curves of each pencil intersect those of one other pencil in three points. The S_3 defined by the curves of such a pair of pencils constitute the two systems of S_3 on an hyperquadric in S_5 which has a double line and contains F.

There are twelve other cubic curves on F. Six are defined by the conics through five simple basis points and the rest by the cubics which have a node at one simple basis point and pass through all the other basis points.

$k\geq 4$.

12. If $k=4$ and if $m=4$, the surface F can be transformed into Π in such a way that the curves Σ_2 correspond to the lines through P', the curves

Σ_1 to quartics with a basis double point P_1 and seven basis simple points $P_2, P_3, \ldots, P_8$, and that the plane sections of F correspond to quartics which have a double point at P_1 and pass through $P_2, P_3, \ldots, P_8$ and P'. Such a linear system defines a quartic F with a double line.

We have seen (Art. 2) that if $k>4$, then $m\leq 3$. Hence, *if* $k\geq 4$, *the surface* F *is a quartic with a double line, or a cubic, a quadric or a plane.*

CASE II. THE CUBICS OF Σ_1 ARE RATIONAL; THOSE OF Σ_2, ELLIPTIC.

$k=1$.

13. The pencil Σ_2 is rational since it is in (1, 1) correspondence with a generic curve of Σ_1. Similarly, Σ_1 is elliptic. Let the pencil Σ_1 be put in (1, 1) correspondence with the pencil of rectilinear generators of a ruled quintic surface ϕ belonging to S_4 and let the pencil Σ_2 be put in (1, 1) correspondence with a pencil Σ' of quartics on ϕ having three basis points P_1, P_2, P_3. Then F and ϕ are in (1, 1) correspondence in such a way that corresponding points are at the intersections of corresponding curves.

14. Denote by ρ_1, ρ_2, ρ_3 the multiplicities at P_1, P_2, P_3 of the curves on ϕ which correspond to the hyperplane sections of F. The curves of this linear system intersect the rectilinear generators in three points and the quartics of Σ' in $3+\rho_1+\rho_2+\rho_3$ points. Their order $6+\rho_1+\rho_2+\rho_3$ is found by counting their intersections with an hyperplane containing a quartic of Σ' and a generator. These curves are cut from ϕ by a linear system of quintic hypersurfaces H^5 which have contact of second order with ϕ at P_1, P_2 and P_3, and contain as basis curves (a), the generators through P_1, P_2, P_3 counted respectively $3-\rho_1$, $3-\rho_2$, $3-\rho_3$ times, (b) a quintic curve C^5 which intersects the generators twice and (c), five generators of ϕ. For, since C^5 is elliptic, in a linear system of seventy-four dimensions * of H^5 that do not contain ϕ, there exists a linear system of thirty-one dimensions of H^5 which contain C^5 and have contact of second order at P_1, P_2 and P_3. The curves corresponding to the hyperplane sections of F intersect these H^5 in $3(\rho_1+\rho_2+\rho_3)$ points at P_1, P_2, P_3; in $2(\rho_1+\rho_2+\rho_3)-3$ points on C^5 and in thirty-three other points. These curves are of genus 3 (at least) since F is neither ruled nor rational,† hence each curve of the system lies on such an H^5. The residual intersection does not intersect a generic generator. Hence it degenerates into the generators through P_1, P_2, P_3 counted $3-\rho_1$, $3-\rho_2$ and $3-\rho_3$ times and five generators.

* Cf. the Author, AMERICAN JOURNAL OF MATHEMATICS, Vol. XLI, p, 51.

† Cf. Castelnuovo–Enriques, *Mathematische Annalen*, Vol. XLVIII, p. 308.

The linear system of curves on ϕ which corresponds to the hyperplane sections of F is thus contained in (or coincides with) a linear system L of curves of order 15 and genus 7, having triple points at P_1, P_2, P_3 which is cut from ϕ by a linear system of H^5 which contain C^5 and five fixed generators, and have contact of second order with ϕ at P_1, P_2, P_3. It follows that: *the surface F is of order* 18, *it belongs to a space of eleven dimensions and its generic hyperplane sections are of genus* 7, *or it is a projection of such a surface.*

15. Let the correspondence between Σ_1 and the generators of ϕ be set up in such a way that the g_3^2 defined on a given generic cubic C of Σ_2 corresponds to the g_3^2 defined on the quartic corresponding to C by the hypercubics having second order contact with ϕ at P_1, P_2, P_3. Then the linear system L is cut from ϕ by this system of hypercubics, that is: *the surface F can be birationally represented on a ruled quintic belonging to S_4 in such a way that the curves corresponding to the hyperplane sections of F are cut from the quintic by cubic hypersurfaces which have second order contact with the quintic at the basis points of a pencil of quartic curves on it.*

16. Let F belong to S_{11}. Its parametric equations can be reduced to the form

$$\begin{array}{llll} x_0=1 & x_1=v & x_2=v^2 & x_3=v^3 \\ x_4=\wp(u) & x_5=v\wp(u) & x_6=v^2\wp(u) & x_7=v^3\wp(u) \\ x_8=\wp'(u) & x_9=v\wp'(u) & x_{10}=v^2\wp'(u) & x_{11}=v^3\wp'(u), \end{array}$$

wherein $\wp(u)$ is the Weierstrassian $\wp$-function.

The residual section of F by an hyperplane which contains the planes of three cubics of Σ_2 degenerates into three curves of Σ_1 since it intersects a generic curve of Σ_2 in three points and has no points in common with a generic curve of Σ_1. Three such cubics of Σ_1 lie in the S_7 defined by the S_3 of two of them, since every hypersurface that contains two of them contains the third.

17. The order of any curve on F that intersects the curves of Σ_1 in x points, and those of Σ_2 in y points is

$$n=3(x+y),$$

as may be seen by counting its intersections with a generic hyperplane which intersects F in three cubics of each system. It follows that the order of every curve on F is a multiple of three. Moreover, since there are no rational curves on F except those of Σ_1, and since every curve C on F for which $y=1$ is in (1, 1) correspondence with Σ_2 and hence is rational, we conclude that *there are no curves of order less than nine on the surface F belonging to S_{11}*

except those of Σ_1 *and* Σ_2. Curves of order 9 on F actually exist. Such a curve is defined by a generic cubic on ϕ, or by a generic quartic through P_1, P_2 or P_3.

$$k=2.$$

18. The involution γ_2^1 of order 2 defined by Σ_1 on a generic cubic C of Σ_2 is rational. For, if γ_2^1 is elliptic, it has no double points, so that no curve of Σ_1 has its two intersections with C coincident. But the involution defined by Σ_2 on a generic curve of Σ_1 has two double points. Hence each of two fixed curves of Σ_2 would degenerate into a curve counted twice. These degenerate curves are of genus 1 since they are in (1, 1) correspondence with Σ_1. But this is impossible, since the curves of Σ_2 are cubics. Then γ_2^1 is rational, Σ_1 is rational, and F is rational since it contains a rational pencil of rational curves.

19. The surface F can be transformed into a plane Π in such a way that to Σ_1 corresponds a pencil of lines through P' and to Σ_2, a pencil of cubics through P' and eight other points $P_1, P_2, \ldots, P_8$. The linear system of curves of order μ corresponding to the hyperplane sections of F can be reduced to a system which coincides with, or is contained in the complete linear system of quartics which pass through P' and have multiplicities ρ_i at P_i such that either

$$(1) \quad \rho_1=\rho_2=\ldots=\rho_8=1,$$

or

$$(2) \quad \rho_1=2,\ \rho_2=\ldots=\rho_7=1,\ \rho_8=0.$$

Hence, if a generic cubic of Σ_1 is rational, and of Σ_2 elliptic, and if $k=2$, then either

(1) *The surface is of order* 7, *it belongs to an* S_5 *and is defined parametrically by the quartics in a plane through the nine basis points of a pencil of cubics, or it is a projection of such a surface*, or (2) *The surface is of order* 5, *it belongs to an* S_4 *and is defined parametrically by the quartics in a plane with a double and seven simple basis points, or it is a projection of such a surface.*

On the surface (1), there are in general eight other pencils of rational cubics which intersect the cubics of Σ_2 in two points and 126 cubics which intersect the cubics of Σ_2 once. On the surface (2) there are in general, sixty-three other pencils of rational cubics which intersect the cubics of Σ_2 twice, and eighty-six cubics which intersect those of Σ_2 once.

$$k=3.$$

20. As in the preceding case, the involution γ_3^1 defined by the curves of Σ_1 on a generic curve of Σ_2 is rational. For, if not, it has no double points,

while the involution defined by Σ_2 on a generic curve of Σ_1 has six double points. Then six curves of Σ_2 have as double component a curve of genus 1, which is impossible.

It follows that F is rational. It can be represented on a plane Π in such a way that Σ_1 corresponds to a pencil of lines through P', Σ_2 to a pencil of cubics through $P_1, P_2, \ldots, P_9$ and the hyperplane sections to a system of curves which coincide with, or are contained in the complete system of quartics which pass through P' and have multiplicities ρ_i at P_i such that either

$$(1) \quad \rho_1=\rho_2=\ldots=\rho_9=1,$$

or

$$(2) \quad \rho_1=2, \ \rho_2=\rho_3=\ldots=\rho_8=1, \ \rho_9=0.$$

Hence, if a generic cubic of Σ_1 is rational and of Σ_2, elliptic, and if $k=3$, then either

(1) *The surface is of order* 6, *belongs to an* S_4 *and is defined parametrically by the quartics through ten points of which nine are the basis points of a pencil of cubics, or it is a projection of such a surface*, or (2) *The surface is a quartic with a nodal line. It belongs to* S_3.

21. Each of these surfaces is the projection of one of the surfaces of Art. 19 from a point P on the surface. Each has a double line l which is the projection of the elliptic cubic through P. The cubics of Σ_1 are the projections of a pencil of quartics through P. On the surface (1), the cubics of Σ_1 have l as a locus of double points. The planes of the cubics of Σ_1 and Σ_2 on this surface in S_4 generate a singular hyperquadric which has F as its complete intersection with an hypercubic.

CASE II. THE CUBICS OF BOTH SYSTEMS ARE ELLIPTIC.

$k=1.$

22. We can write at once the parametric equations of a surface which clearly belongs to the required type. Let $\wp(u)$ and $\bar{\wp}(v)$ be two Weierstrassian $\wp$-functions (having in general different moduli). Then the surface

$$\left.\begin{array}{lll} x_0=1 & x_1=\wp(u) & x_2=\wp'(u) \\ x_3=\bar{\wp}(v) & x_4=\wp(u)\bar{\wp}(v) & x_5=\wp'(u)\bar{\wp}(v) \\ x_6=\bar{\wp}'(v) & x_7=\wp(u)\bar{\wp}'(v) & x_8=\wp'(u)\bar{\wp}'(v) \end{array}\right\} \qquad (1)$$

is generated by two pencils of elliptic cubics $u=$const. and $v=$const. such that cubics of opposite systems intersect in one point.

23. Any two cubics of one system on (1) lie in an S_5 which contains a third cubic of the same system. The residual section of the surface by an

hyperplane that contains such an S_5 is composed of three cubics of the opposite system. Hence, the order of this surface is 18 and, since cubics of the same system do not intersect, the genus of a generic hyperplane section is equal to 10.* It follows further that a curve on the surface which intersects the curves u=const. in x-points and the curves v=const. in y-points is of order $n=3(x+y)$ as may be seen by counting the intersections of the curve with an hyperplane which intersects (1) in six cubics. If either $x=1$, $y>0$ or $y=1$, $x>0$ there is a $(1, y)$ or $(1, x)$ correspondence between the two systems of cubics, and hence a particular relation between the moduli. Hence, *in general there are no curves of order less than* 12 *on the surface* (1) *except the cubics u=const. and v=const.*

24. Let F be a given surface generated by two pencils Σ_1 and Σ_2 of elliptic cubic curves that intersect in one point. Let the moduli of the pencils on (1) be respectively equal to those of Σ_1 and Σ_2 and let the pencils u=const. and Σ_1 and v=const. and Σ_2 be put in (1, 1) correspondence. Then the surfaces (1) and F are in (1, 1) correspondence in such a way that corresponding points are at the intersection of corresponding curves. To an hyperplane section of F corresponds on (1) a curve C^{18} which is of order 18 since it intersects the curves u=const. and v=const. in three points.

25. In the S_8 defined by the surface (1), there exists a linear system of $9n^2-1$ dimensions of hypersurfaces H^n of order n which do not contain the surface (1) since such an H^n can be made to contain $3n-1$ generic curves u=const. and $3n-1$ generic points of an additional curve u=const. without containing (1). If n is sufficiently large, there are, in such a linear system, at least $3n-12$ linearly independent H^n which contain a given C^{18} and $3n-4$ given generic cubics u=const. The residual intersection of all these H^n with any given generic cubic v=const. is a fixed point P. Hence they all contain a fixed curve C on (1) which intersects the cubics v=const. in one point and the cubics u=const. in a certain number $y\geq 0$ of points. Their residual intersections with (1) have no points in common with a generic curve v=const. and break up into $3n-3-y$ curves of that pencil. The curve C is of order $3(1+y)$. Its genus is unity since it intersects the curves v=const. once. In general, $y=0$ (Art. 23).

26. If $y=0$, the curve C is a cubic of u=const. and the curves C^{18} corresponding to the hyperplane sections of F are defined on (1) by a linear system of H^n through $3n-3$ fixed cubics of each system. The complete linear system to which they belong is of dimension $9n^2-1-[3n(3n-3)+3(3n-3)]=8$.

* Cf. Castelnuovo-Enriques, *Annale de Matematica*, Ser. 3, Vol. VI, p. 171.

Let

$$u_1, u_2, \ldots, u_{3n-3} \text{ and } v_1, v_2, \ldots, v_{3n-3}$$

be the parameters of the $6n-6$ basis cubics of the given system of H^n. The parameters u, u', u'' of the three intersections with any cubic $v=$const. of a C^{18} corresponding to an hyperplane section of F satisfy the congruence

$$u+u'+u'' \equiv -(u_1+u_2+\ldots+u_{3n-3}) \quad \text{mod } (\omega_1\omega_2).$$

Similarly, the parameters v, v', v'' of its intersections with a cubic $u=$const. satisfy

$$v+v'+v'' \equiv -(v_1+v_2+\ldots+v_{3n-3}) \quad \text{mod } (\overline{\omega}_1\overline{\omega}_2).$$

Let the correspondence between F and (1) be set up in such a way that the g_3^2 defined by the lines in the plane of one given cubic of each system on F is transformed into the g_3^2 defined by the lines in the plane of the corresponding cubic on (1). Then each member of the above congruences is congruent to zero, and the complete linear system of eight dimensions, to which the curves C^{18} belong is cut from (1) by a linear system of H^n which degenerate into a fixed H^{n-1} containing the $6n-6$ cubics, together with the system of hyperplanes of S_8. We conclude that *in general*, $y=0$ *and the surface* F *coincides with a surface* (1) *or with a projection of such a surface.*

27. If $y>0$, the curves corresponding to the hyperplane sections of F are cut from (1) by a linear system of H^n which contain $3n-4$ fixed cubics $u=$const., $3n-3-y$ fixed cubics $v=$const., and a fixed curve C of order $3(1+y)$ and genus 1. The complete linear system L of curves cut from (1) by the H^n which contain the above curves is of dimension

$$9n^2-1-[3(1+y)n+(3n-4)(3n-y)+(3n-3-y)3]=8-y.$$

Since the dimension of this system can not be less than 3, we have $y\leq 5$.

The curve C intersects the curves of the system L in $3+5y$ points. In fact, by varying the basis curves $u=$const. we can obtain a pencil of curves C (no two of which intersect) each of which, with fixed curves $u=$const. and $v=$const., forms the system of basis curves of a linear system of H^n that define L. Each curve C intersects an H^n of a system which does not contain it in $3n(1+y)$ points of which $3n(1+y)-5y-3$ lie on the basis cubics and $5y+3$ on a curve of L. It follows that the order of the system L and hence of the surface F, is

$$m=18n-(5y+3)-3(6n-7-y)=18-2y.$$

Let p be the genus of a generic curve of L. The virtual genus of the residual section of (1) by an H^n of a linear system which defines L is

$9(n-1)^2-y+1$, and of the complete section of (1) by H^n is $9n^2+1$. Then

$$p+9(n-1)^2-y+1+18n-(18-2y)-1=9n^2+1$$

or

$$p=10-y.$$

Hence, *if* $5\geq y\geq 0$, *the surface* F *is of order* $18-2y$, *its hyperplane sections are of genus* $10-y$ *and it belongs to a space of* $8-y$ *dimensions, or it is a projection of such a surface.*

$$k=2.$$

28. If F belongs to an $S_r(r>3)$ let it be projected from an S_{r-4} which does not intersect it onto a surface F' belonging to S_3. Denote the projections of Σ_1 and Σ_2 by Σ_1' and Σ_2'. If the involution γ_2^1 defined on a generic curve C' of Σ_1' by the curves of Σ_2' is elliptic, the lines joining corresponding points envelope a curve of class 3, if it is rational, they constitute a pencil with vertex on C'. Since at most one cubic of Σ_2 can be coplanar with C'—otherwise a generic cubic of Σ' would have four points in common with this plane—the surface F' belongs to one of the following types:

α. The curves of Σ_1' and Σ_2' lie in pairs in the tangent planes of a developable of class 4 and genus 1. This developable can not reduce to a cone since it can not have a double plane.

β. The planes of each system envelope a cone of class 3 and genus 1.

γ. The planes of Σ_1' generate an axial pencil; those of Σ_2' envelope a cone of class 3 and genus 1.

δ. The planes of each system generate an axial pencil.

ε. The curves of Σ_1' and Σ_2' lie in pairs in the tangent planes to a quadric cone.

29. Cases α, β and γ do not exist. In case α, the residual section of F' by a plane of the developable is elliptic since it is intersected by the curves of Σ_1 and Σ_2 in one variable point. It is a cubic since the order of F' does not exceed 9 (Art. 2). Denote this system of cubics by Σ_3'. Two generic cubics of Σ_3' intersect in one variable point since, of the three intersections of each with the plane of the other, only two lie on the curves of Σ_1' and Σ_2'. Hence F' contains a pencil of rational curves* on which the pencil Σ_1' (or Σ_2') defines an elliptic involution. This is impossible.

In case β, let P_1 and P_2 be the vertices of the cones. The planes of three curves of Σ_2' pass through P_1. The system of lines joining corresponding points of the involution on each of these cubics degenerates into a pencil counted three times. Then these curves degenerate and intersect a generic

* Cf. the Author, AMERICAN JOURNAL OF MATHEMATICS, Vol. XLI.

curve of Σ_1 in coincident points. But this is impossible since the involution on a generic curve of S_1 has no double points.

In case γ, let μ be the multiplicity on F' of the axis l of the pencil of planes of Σ_1'. Then the order of F' is $\mu+3$. A generic curve C' of Σ_2' intersects l. The residual section in its plane is of order μ and has a $(\mu-1)$-fold point on l. Hence it is rational. The curves of Σ_2' define an elliptic involution on this rational curve, which is impossible. We now conclude that Σ_1' and Σ_2' are both rational.

30. In case δ, we denote the axes of the pencils by l_1 and l_2 and suppose, first, that they do not intersect. Then l_1 (or l_2) is a simple line on the surface since a generic point of it is a simple point on just one curve of Σ_2' (or Σ_1'). Hence F' is a quartic with two skew simple lines. Since l_1 and l_2 do not intersect, and the cubics of Σ_1 and Σ_2 are elliptic, the quartic F is not the projection of a surface of the same order belonging to more than three dimensions; for, the genus of a generic plane section is less than 2 only if the surface is a ruled quartic of genus 1.

31. If l_1 and l_2 intersect, all the curves of both systems pass through their intersection. Let μ be the multiplicity of l_1 and l_2 on F'. Then one curve of each system degenerates into the axis of the other system counted μ times, together with a residual curve of order $3-\mu$. Thus $\mu\leq 3$ and $m=3+\mu\leq 6$. The intersection of l_1 and l_2 is a $(\mu+1)$-fold point on the surface. The linear system of quadrics which contain l_1 and l_2 defines a birational transformation of F' into a sextic surface F_1 belonging to S_4 and of Σ_1' and Σ_2' into pencils of cubics on F_1, all of which pass through a fixed point P which is a double point on F_1. The planes of these two pencils of cubics constitute the two systems of planes on an hyperquadric H^2 with a double point at P. The surface F_1 lies on (at least) six linearly independent hypercubics of which at most five have H^2 as a component. Hence, it forms the complete intersection of H^2 with an hypercubic. The surface F' is the projection of F_1 from a point on H^2.

32. In case ε, all the cubics of both systems pass through the vertex of the cone which is a double point on F'. The order of F' is 6 since a residual curve in the plane of two coplanar cubics would have to be a component of a curve of each system. Three of the intersections of coplanar cubics lie on a generator of the quadric cone. The other six lie on a nodal sextic which forms the intersection of a cubic and a quadric surface. The linear system of cubic surfaces which contain this nodal sextic define a birational transformation of

F' into a surface of the type of F_1 of Art. 31. The surface F' is the projection of the surface F_1 from a point not lying on the hyperquadric.

33. Every surface F belonging to S_4 is of the type F_1 since it lies on, at most, one hyperquadric, and its projection from a point not on this hyperquadric is a sextic F' of the type of Art. 32. Since no surface F_1 is a projection of a surface of the same order belonging to S_5, we conclude that, if generic cubics of Σ_1 and Σ_2 are elliptic and if $k=2$, then

(1) *If the cubics of both systems do not all pass through a fixed point P, the surface is a quartic, belongs to S_3 and has two skew simple lines.*

(2) *If the cubics of both systems pass through a fixed point P, the surface is the complete intersection of a cubic hypersurface in S_4 with a quadric hypersurface which has a double point at P, or it is a projection of such a surface.*

34. Neither of these surfaces contains, in general, any other pencils of cubics. In fact, their geometric genus is in general unity so that they do not contain a pencil of rational cubics. If the surface (1) contains an additional pencil of elliptic cubics, it contains an additional simple line, if the surface (2) belonging to S_4 contained such a pencil, the planes of the cubics would lie on the hyperquadric.

$k=3$.

35. If F belongs to an S_r $(r>3)$ let it be projected from an S_{r-4} that does not intersect it into a surface F' belonging to S_3. The lines of section of the plane of a cubic C' of Σ_1' by the the planes of Σ_2' constitute a pencil since through a generic point of C' there passes just one such line. Hence, either the planes of Σ_2' (and, similarly, of Σ_1') constitute an axial pencil or the curves of the two systems lie in pairs in the tangent planes to a quadric cone.

36. Let the two systems of planes constitute two pencils whose axes l_1 and l_2 do not intersect. Then neither l_1 nor l_2 lies on F' so that F' is a cubic. Since a generic curve of Σ_1 (or Σ_2) is elliptic, this cubic can not have a nodal line and is not the projection of a surface of the same order belonging to S_4.

37. If the axes of the pencils intersect, or if the curves lie in pairs in the tangent planes to a quadric cone, we see as in Arts. 31 and 32, that F' is the projection of a surface F_1 belonging to S_4 which forms the complete intersection of a cubic hypersurface with a quadric hypersurface having a double point which does not lie on the cubic.

38. It follows as in Art. 33, that if generic cubics of Σ_1 and Σ_2 are elliptic and if $k=3$, then

(1) *If the planes of the cubics of both systems do not all pass through the same fixed point, the surface is a cubic which can not have a double line.*

(2) *If the planes of the cubics of both systems pass through the same fixed point, the surface is the complete intersection of a cubic hypersurface in S_4 with a quadric hypersurface which has a double point not lying on the cubic, or it is a projection of such a surface.*

As in Art. 34, it is seen that the surface (2) does not in general contain any other pencils of cubic curves.

39. The results of the foregoing discussion are summarized in the following table. In this table p_1 and p_2 denote respectively the genus of a generic curve of Σ_1 and Σ_2, k is the number of variable intersections of curves of opposite systems; m is the maximum order of a surface F containing Σ_1 and Σ_2; p is the maximum genus of an hyperplane section of F; r is the maximum number of dimensions to which F belongs, and s is the number of distinct or consecutive pencils of cubic curves necessarily existing on the surface F of maximum order. The numbers p_g and p_a are respectively the geometric and arithmetic genus of F except in the last two cases, where they are the maximum values of those numbers.

p_1	p_2	k	m	r	p	s	p_g	p_a
0	0	1	18	15	4	2	0	0
0	0	2	8	7	2	8	0	0
0	0	3	6	5	2	32	0	0
0	0	4	4	3	2	128	0	0
0	1	1	18	11	7	2	0	—1
0	1	2	7	5	3	10	0	0
0	1	3	6	4	3	11	0	0
1	1	1	18	8	10	2	1	—1
1	1	2	6	4	4	2	1	1
1	1	3	6	4	4	2	1	1

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